Lecture 14: Examples of Martingales and Azuma's Inequality

A Short Summary of Bounds I

- Chernoff (First Bound). Let $\mathbb{X}$ be a random variable over $\{0,1\}$ such that $\mathbb{P}[\mathbb{X}=1]=p$ and $\mathbb{P}[\mathbb{X}=0]=1-p$.

$$\mathbb{P}\left[\sum_{i=1}^{n}\mathbb{X}^{(i)}-np\geqslant t\right]\leqslant \exp\left(-n\mathrm{D}_{\mathrm{KL}}\left(p+t/n,p\right)\right)$$

- Azuma, Hoeffding, Chernoff (Second Bound). For a martingale difference sequence $(\Delta\mathbb{F}_1,\ldots,\Delta\mathbb{F}_n)$ such that $\mathbb{F}_i$ takes values in the range $[a_i,b_i]$. Then, we have

$$\mathbb{P}\left[\sum_{i=1}^{n}\Delta\mathbb{F}_i\geqslant t\right]\leqslant \exp(-\frac{2t^2}{\sum_{i=1}^{n}(b_i-a_i)^2})$$

A Short Summary of Bounds II

- Talargrand. Let $\mathbb{X} = (\mathbb{X}_1, \ldots, \mathbb{X}_n)$ be independent variables. Talagrand inequality states that

$$\mathbb{P}\left[\mathbb{X} \in A\right] \mathbb{P}\left[d_T(\mathbb{X}, A) \geqslant t\right] \leqslant \exp(-t^2/4)$$

  We can use this to show concentration of a configuration function $f(\mathbb{X}_1, \ldots, \mathbb{X}_n)$ around its median.

Experiment.

- There are N balls in a box. Among these balls, at time $t = 0$, there are $R = pN$ red balls, and $B = (1 - p)N$ blue balls
- At any time, we draw a random ball from the box (and we do not replace the ball back in the box).
- We are interested in understanding the concentration of the random variable representing the total number of red balls seen at the end of time n.
- We assume that $N \gg n$, i.e. the bin never runs out of balls in our experiment

Formalization.

- We shall represent a red balls by 1, and a blue ball by 0
- The variables $(\mathbb{X}_1, \dots, \mathbb{X}_n)$ represent the balls we draw at time $1, \dots, n$, respectively
- We are interested in understanding the concentration of the random variable $\mathbb{S}_n = \sum_{i=1}^{n} \mathbb{X}_i$. Note that the probability of $\mathbb{X}_i = 1$ depends on the sum $\mathbb{S}_{i-1} = \sum_{j=1}^{i-1} \mathbb{X}_j$.
- By linearity of expectation we had already concluded that

$$\mathbb{E}\mathbb{S}_n = np$$

Constructing a Martingale.

- Suppose we have already seen $\mathbb{X}_1 = \omega_1, \dots, \mathbb{X}_k = \omega_k$
- The total number of red balls in the box is $R' = Np - \mathbb{S}_k$, and the total number of blue balls in the box at this time is $B' = N(1-p) - k + \mathbb{S}_k$. Recall that we have $\mathbb{S}_k = \sum_{i=1}^{k} \mathbb{X}_i$.
- Then, the expected number of red balls seen in the future is

$$(n-k)\frac{Np - \mathbb{S}_k}{N-k}$$

- Let us define the random variable, for $k \in \{0, \dots, n\}$,

$$\mathbb{F}_k = \mathbb{S}_k + (n-k)\frac{Np - \mathbb{S}_k}{N-k}$$

Hypergeometric Series IV

- $(\mathbb{F}_0, \ldots, \mathbb{F}_n)$ is the Doob's martingale corresponding to the function

$$f(\mathbb{X}_1, \ldots, \mathbb{X}_n) = \sum_{i=1}^{n} \mathbb{X}_i$$

- Note that $\mathbb{F}_n = \mathbb{S}_n$ and $\mathbb{F}_0 = np$
- Let the martingale difference sequence corresponding to this martingale be $(\Delta\mathbb{F}_1, \ldots, \Delta\mathbb{F}_n)$
- In this martingale difference sequence, we have

$$(b_i - a_i) = 1 - \frac{n-i}{N-i} = \frac{N-n}{N-i} \leqslant 1$$

Hypergeometric Series V

- Azuma's inequality on the corresponding martingale difference sequence yields

$$\mathbb{P}\left[\mathbb{S}_n - np \geqslant t\right] \leqslant \exp\left(-\frac{2t^2}{\sum_{i=1}^{n}\left(\frac{N-n}{N-i}\right)^2}\right)$$
$$\leqslant \exp\left(-\frac{2t^2}{n}\right)$$

Note that $\mathbb{E}\left[\mathbb{S}_n\right] = np$.

Experiment.

- There are N balls in an urn. Among these balls, at time $t = 0$, there are R red balls, and $B = (N - R)$ blue balls
- At any time, we draw a random ball from the urn. If the color of the ball is red, then we replace the ball and add one new red ball to the urn. If the color of the ball is blue, then we replace the ball and add one new blue ball to the urn.
- We are interested in understanding the concentration of the random variable representing the total number of red balls seen at the end of time n.

Formalization.

- We shall represent a red ball by 1, and a blue ball by 0.
- The variables $(\mathbb{X}_1, \ldots, \mathbb{X}_n)$ represent the balls we draw at time $1, \ldots, n$, respectively.
- We are interested in understanding the concentration of the random variable $\mathbb{S}_n := \sum_{i=1}^{n} \mathbb{X}_i$. Note that the probability of $\mathbb{X}_i = 1$ depends on the sum $\mathbb{S}_{i-1} = \sum_{j=1}^{i-1} \mathbb{X}_j$.

Póly's Urn III

Lemma

$$\mathbb{E}\left[\mathbb{S}_n\right] = n\frac{R}{R+B}$$

The proof of this theorem using induction on n is left as an easy exercise.

Constructing a Martingale.

- Suppose we have already seen $\mathbb{X}_1 = \omega_1, \ldots, \mathbb{X}_k = \omega_k$
- The total number of red balls in the urn at this time is $R' = R + \sum_{i=1}^{k} \mathbb{X}_i$, and the total number of blue balls in the urn at this time is $B' = B + k - \sum_{i=1}^{k} \mathbb{X}_i$. Recall that we have $\mathbb{S}_k = \sum_{i=1}^{k} \mathbb{X}_i$.
- Then, the expected number of red balls seen in the future is

$$(n-k)\frac{R'}{R'+B'}$$

- Let us define the random variable, for all $k \in \{0, \ldots, n\}$,

$$\mathbb{F}_k = \mathbb{S}_k + (n-k)\frac{R+\mathbb{S}_k}{N+k}$$

Póly's Urn V

- $(\mathbb{F}_0, \ldots, \mathbb{F}_n)$ is the Doob's martingale corresponding to the function

$$f(\mathbb{X}_1, \ldots, \mathbb{X}_n) = \sum_{i=1}^{n} \mathbb{X}_i$$

- Note that $\mathbb{F}_n = \mathbb{S}_n$ and $\mathbb{F}_0 = n\frac{R}{N}$
- Let the martingale difference sequence corresponding to this martingale be $(\Delta\mathbb{F}_1, \ldots, \Delta\mathbb{F}_n)$
- In this martingale difference sequence, we have

$$(b_i - a_i) = 1 + \frac{n-i}{N+i} = \frac{N+n}{N+i}$$

- Azuma's inequality on the corresponding martingale difference sequence yields

$$\mathbb{P}\left[\mathbb{S}_n - n\frac{R}{N} \geqslant t\right] \leqslant \exp\left(-\frac{2t^2}{\sum_{i=1}^{n}\left(\frac{N+n}{N+i}\right)^2}\right)$$

$$\leqslant \exp\left(-\frac{2t^2}{n(1+n/N)}\right)$$

Note that $\mathbb{E}\left[\mathbb{S}_n\right] = n\frac{R}{N}$.